CHAPTER-3

Complete Metric Spaces.

Definition: A sequence $\{x_n\}$ in a metric space (X, d) is said to converge to an element $u \in X$ if $d(x_n, u) \to 0$ as $n \to \infty$.

In other words, a sequence $\{x_n\}$ is said to converge to an element $u \in X$ if for a given $\epsilon > 0$, $\exists$ a positive integer n_0 such that

$$d(x_n, u) < \epsilon \quad \forall n \geq n_0, \text{ i.e.,}$$

$$x_n \in S_\epsilon(u), \; \forall n \geq n_0.$$

We then write $\lim_{n \to \infty} x_n = u$

Theorem: In a metric space a convergent sequence cannot have distinct limits.

proof: Let $\{x_n\}$ be a convergent sequence in a metric space (X, d). If possible, let $\{x_n\}$ has two distinct limits ~~x~~ x and y. Then due to T_2 property of (X, d) $\exists$ an $\epsilon_0 > 0$ such that $S_{\epsilon_0}(x) \cap S_{\epsilon_0}(y) = \phi$

Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ $\exists$ two positive integer n_1 and n_2 such that

$$x_n \in S_{\epsilon_0}(x), \forall n \geq n_1 \text{ and}$$
$$x_n \in S_{\epsilon_0}(y), \forall n \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then $x_n \in S_{\epsilon_0}(x) \cap S_{\epsilon_0}(y)$, $\forall n \geq n_0$ which contradicts the fact that $S_{\epsilon_0}(x) \cap S_{\epsilon_0}(y) = \phi$.

Therefore, a convergent sequence in a metric space cannot have ~~two~~ distinct limits.

Definition:- A sequence $\{x_n\}$ in a metric space (X, d) is called cauchy sequence if for a preassigned $\epsilon > 0$ $\exists$ a positive integer n_0 such that $d(x_m, x_n) < \epsilon$, $\forall$ ~~$n \geq$~~ $m, n \geq n_0$.

Theorem:- Every Convergent sequence in a metric space is a cauchy sequence but not conversely.

proof: Let $\{x_n\}$ be a convergent sequence in a metric space (X, d) with $\lim_{n\to\infty} x_n = l$. Let $\epsilon > 0$ be preassigned. Then $\exists$ a positive integer n_0 such that

$$d(x_n, l) < \epsilon/2, \quad \forall n \geq n_0.$$

So, for $m, n \geq n_0$; we have

$$d(x_m, x_n) \leq d(x_m, l) + d(l, x_n)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore $\{x_n\}$ is a cauchy sequence.

However the converse ~~is~~ of this theorem is not true. For example, let us consider $X = (0, 1)$ as a metric space with respect to usual metric d. Then $\{x_n\} = \{\frac{1}{n}\}$, $n \geq 1$ is a cauchy sequence in (X, d) because $d(x_m, x_n) = d(\frac{1}{m}, \frac{1}{n}) = |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n} \to 0$ as $m, n \to \infty$. But $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0 \notin X$. Therefore, $\{\frac{1}{n}\}, n \geq 1$ is a cauchy sequence in (X, d) without being convergent in (X, d).

Definition:- A metric space (X, d) is said to be complete if every cauchy sequence in (X, d) is a convergent sequence in (X, d) and it is said to be incomplete if $\exists$ at least one cauchy sequence in (X, d) which is not convergent in (X, d).

Example: The Euclidian n-space $\mathbb{R}^n$ is a complete metric space w.r.to the usual metric on $\mathbb{R}^n$.

Sol^n: Let $\{x_r\}$ be a cauchy sequence in $\mathbb{R}^n$, where

$$x_r = (\xi_1^{(r)}, \xi_2^{(r)}, \dots, \xi_i^{(r)}, \dots \xi_n^{(r)}).$$

Let 'd' be the usual metric on $\mathbb{R}^n$. Since $\{x_r\}$ is a cauchy sequence in $(\mathbb{R}^n, d)$ $d(x_r, x_s) \to 0$ as $r, s \to \infty$, i.e;

$$\sqrt{\sum_{i=1}^{n} (\xi_i^{(r)} - \xi_i^{(s)})^2} \to 0 \text{ as } r,s \to \infty \text{ i.e;}$$

$$\sum_{i=1}^{n} (\xi_i^{(r)} - \xi_i^{(s)})^2 \to 0 \text{ as } r, s \to \infty.$$

$$(\xi_i^{(r)} - \xi_i^{(s)})^2 \to 0 \text{ as } r, s \to \infty \text{ for each } i = 1, 2, \dots, n.$$

So, $$|\xi_i^{(r)} - \xi_i^{(s)}| \to 0 \text{ as } r, s \to \infty \text{ for each } i = 1, 2, \dots, n$$

This shows that $\{\xi_i^{(r)}\}$ is a cauchy sequence for each $i = 1, 2, \dots, n$.

Therefore by ~~as~~ cauchy general principle of Convergence $\{\xi_i^{(r)}\}$ converges for each $i = 1, 2, \dots, n$.

Let $\lim_{r\to\infty} \xi_i^{(r)} = \xi_i^{(0)}$ for $i = 1, 2, \dots, n$.

Let $x_0 = (\xi_1^{(0)}, \xi_2^{(0)}, \dots, \xi_n^{(0)})$. Then $x_0 \in \mathbb{R}^n$.

Since, $d(x_r, x_s) \to 0$ as $r, s \to \infty$

taking limit as $s \to \infty$ and keeping r fixed, we have $d(x_r, x_0) \to 0$ as $r \to \infty$, i.e,

$$\lim_{r\to\infty} x_r = x_0 \in \mathbb{R}^n.$$

Thus Every cauchy sequence in $(\mathbb{R}^n, d)$ is a Convergent sequence and so $(\mathbb{R}^n, d)$ is a complete metric space.

Theorem: A Cauchy sequence in a metric space which

$d(x_{n_k}, l) < \epsilon/2$, $\forall k \geq n_2$, $(\because n_k \geq k)$

Let $m = \max\{n_1, n_2\}$. Then for all $n \geq m$, we have

$$d(x_n, l) \leq d(x_n, x_m) + d(x_m, l)$$
$$< \epsilon/2 + \epsilon/2$$
$$= \epsilon$$

Thus $\{x_n\}$ is convergent and this proves the theorem.

Theorem:- Let (X, d) be a metric space and E be a subset of X. Then $\text{diam}(E) = \text{diam}(\bar{E})$

proof: Since $E \subseteq \bar{E}$, we have

$$\sup_{x,y \in E} d(x,y) \leq \sup_{u,v \in \bar{E}} d(u,v)$$

i.e., $\text{diam}(E) \leq \text{diam}(\bar{E})$. ---(1)

In case $\text{diam}(E) = \infty$, $\text{diam}(\bar{E}) = \infty$ and so

$$\text{diam}(E) = \text{diam}(\bar{E})$$

So suppose that $\text{diam}(E) < \infty$.

Let $a', b' \in \bar{E}$ and $\epsilon > 0$ be given. Then $\exists$ two points $a, b \in E$ such that $d(a, a') < \epsilon/2$ and $d(b, b') < \epsilon/2$

Now, $d(a', b') \leq d(a', a) + d(a, b')$

$$\leq d(a', a) + d(a, b) + d(b, b')$$
$$< \epsilon/2 + \epsilon/2 + d(a, b)$$
$$= \epsilon + d(a, b)$$
$$\leq \epsilon + \text{diam}(E)$$

Since the right side of the above inequality is independent of the case of a', b' therefore

$$\sup_{a', b' \in \bar{E}} d(a', b') \leq \epsilon + \text{diam}(E)$$

i.e., $\text{diam}(\bar{E}) \leq \epsilon + \text{diam}(E)$

Since $\epsilon > 0$ is arbitrary, $\text{diam}(\bar{E}) \leq \text{diam}(E)$ ---(2)

Thus by (1) & (2), we have

$$\text{diam}(E) = \text{diam}(\bar{E}).$$

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Theorem: (Cantor's Intersection theorem).

A necessary and sufficient condition that a metric space (X,d) to be complete is that for every decreasing sequence $\{F_n : n \in N\}$ of non-empty closed subsets with $\delta(F_n) \to 0$ as $n \to \infty$, there is one and only one point in $\bigcap_{n=1}^{\infty} F_n$.

proof: Let (X,d) be a complete metric space and $\{F_n : n \in N\}$ be a decreasing sequence of non-empty closed subsets in (X,d) with $\delta(F_n) \to 0$ as $n \to \infty$.

Let $a_n \in F_n$, then $a_{n+p} \in F_{n+p} \subseteq F_n$. So, $a_n, a_{n+p} \in F_n$ and therefore $d(a_n, a_{n+p}) \leq \delta(F_n) \to 0$ as $n \to \infty$. This shows that $\{a_n\}$ is a cauchy sequence in (X,d). Since (X,d) is complete $\{a_n\}$ is convergent. Hence there exist a point $u \in X$ such that $\lim_{n\to\infty} a_n = u$. Let n be a fixed positive integer. Then $a_{n+p} \in F_n$ for $p = 1,2,\ldots n$.

Taking limit as $p \to \infty$, we have

$\lim_{p\to\infty} a_{n+p} = u \in F_n$, Since F_n is closed in (X,d). Let now n be free. Then $u \in F_n$ for $n = 1,2,3,\ldots$ i.e; $u \in \bigcap_{n=1}^{\infty} F_n$.

If possible let $\exists\, v \in X$ such that $v \in \bigcap_{n=1}^{\infty} F_n$. Then $d(u,v) \leq \delta(F_n) \to 0$ as $n \to \infty$. Thus, $d(u,v) = 0$, i.e; $u = v$. Hence $\bigcap_{n=1}^{\infty} F_n$ is a singleton.

Conversely, let the condition of the theorem be satisfied in (X,d). Let $\{a_n\}$ be a cauchy sequence in (X,d)

put $H_n = \{a_n, a_{n+1}, \ldots\}$, $n = 1,2,\ldots$

Then $H_{n+1} \subseteq H_n$ for $n \in N$ and so $\overline{H}_{n+1} \subseteq \overline{H}_n$ for $n \in N$. Since $\{a_n\}$ is a cauchy sequence $d(a_n, a_m) \to 0$ as $n, m \to \infty$. Therefore $\delta(H_n) \to 0$ as $n \to \infty$.

But $\delta(H_n) = \delta(\overline{H}_n)$ for $n = 1,2,3\ldots$ and so $\delta(\overline{H}_n) \to 0$ as $n \to \infty$. Therefore $\{\overline{H}_n : n \in N\}$ is a decreasing sequence of non-empty closed subset in (X,d) with $\delta(\overline{H}_n) \to 0$ as $n \to \infty$. Therefore, by hypothesis $\bigcap_{n=1}^{\infty} \overline{H}_n$ is a singleton, say $\{u\}$. Now, $\{a_n, a_{n+1}, \ldots\} = H_n \subseteq \overline{H}_n$ and $u \in \overline{H}_n$.

Therefore $d(a_n, u) \leq \delta(\overline{H}_n) \to 0$ as $n \to \infty$.

This shows that $\{a_n\}$ converges to $u \in X$.

Hence every cauchy sequence in (X,d) is convergent and consequently (X,d) is a complete metric space.

This completes the proof of the theorem.

Definition: A non empty subset E of a metric space (X,d) is said to be every where dense or simply dense in (X,d) if $\bar{E} = X$.

Clearly the set Q of all rational numbers in the space R of real number ~~is a metric~~ w.r.to usual metric is everywhere dense.

Definition: A subset E of a metric space (X,d) is said to be nowhere dense in (X,d) if $\bar{E}$ has got no interior.

Clearly, the set N of all natural number is nowhere dense in the space R of real numbers with respect to usual metric.

Example: The space $C[a,b]$ of all real valued continuous function over the closed interval $[a,b]$ is a complete metric space with respect to the sup metric 'ρ' defined by

$$\rho(f,g) = \sup_{a \le t \le b} |f(t) - g(t)|, \quad f, g \in C[a,b].$$

Solution: First we show that the convergence of a sequence $\{f_n\}$ of $C[a,b]$ is the same as the uniform convergence of $\{f_n\}$ to f_0. Let ~~sup~~ $\rho(f_n, f_0) \to 0$ as $n \to \infty$.

Then $\sup_{a \le t \le b} |f_n(t) - f_0(t)| \to 0$ as $n \to \infty$ and so $|f_n(t) - f_0(t)| \le \sup_{a \le t \le b} |f_n(t) - f_0(t)| \to 0$ as $n \to \infty$. for $a \le t \le b$. This shows that $\{f_n\}$ converges uniformly to f_0 in $[a,b]$. The converse also true by proceeding the steps backward.

Let $\{f_n\}$ be a cauchy sequence in $C[a,b]$. Then

$$\rho(f_n, f_m) \to 0 \text{ as } n,m \to \infty \text{ ------(1)}$$

So, $\sup |f_n(t) - f_m(t)| \to 0$ as $m,n \to \infty$ for $a \le t \le b$.

Then $|f_n(t) - f_m(t)| \to 0$ as $n,m \to \infty$ for each t in $[a,b]$.

This shows that for a fixed t in $[a,b]$. $\{f_n(t)\}$ is a cauchy sequence of real numbers and therefore by cauchy general principle of convergence $\{f_n(t)\}$ converges.

Let $\lim_{n\to\infty} f_n(t) = f_0(t)$, $a \le t \le b$

~~Let~~ n ~~be fixed~~ Keeping n uchanged and passing taking $m \to \infty$ from (1), we have

$$\rho(f_n, f_0) \to 0 \text{ as } n \to \infty.$$

Therefore $\{f_n\}$ converges uniformly to f_0 ~~in [a,b]~~.

Since $f_n \to f_0$ uniformly as $n \to \infty$ and each f_n is continuous, therefore by Weierstrass's theorem f_0 is continuous in $[a,b]$, i.e.; $f_0 \in C[a,b]$. Thus $\{f_n\}$ converges to $f_0 \in C[a,b]$.

Therefore $C[a,b]$ is a complete metric space.